高等结构动力学

——之第八讲

结构随机振动分析

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1.谱分析基础

1.1 谱密度与相关函数

功率谱：$S_x(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_x(\tau)e^{-i\omega\tau}d\tau=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_x(\tau)\cos\omega\tau d\tau$

相关函数：$R_x(\tau)=\int_{-\infty}^{\infty}S_x(\omega)e^{i\omega\tau}d\omega=\int_{-\infty}^{\infty}S_x(\omega)\cos\omega\tau d\omega$

1.2 互谱密度与互相关函数

互谱密度：$S_{xy}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_{xy}(\tau)e^{-i\omega\tau}d\tau$

互相关函数：$R_{xy}(t_1,t_2)=E[x(t_1)y(t_2)]$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(t_1)y(t_2)f_{xy}[x(t_1),y(t_2)]dxdy$$

1.谱分析基础(续)

1.3 随机过程导数

$$R_{x\dot{x}}(\tau)=E\left[x(t)\dot{x}(t+\tau)\right]=\frac{dR_x(\tau)}{d\tau}$$

$$R_{\dot{x}x}(\tau)=E\left[\dot{x}(t)x(t+\tau)\right]=-\frac{dR_x(\tau)}{d\tau}$$

$$R_{\dot{x}}(\tau)=E\left[\dot{x}(t)\dot{x}(t+\tau)\right]=-\frac{d^2R_x(\tau)}{d\tau^2}$$

$$S_{\dot{x}}(\omega)=\omega^2S_x(\omega)$$

1.4 谱参数

原点矩： $\alpha_j=\int_0^\infty \omega^j S(\omega)d\omega$

形状参数： $q=\sqrt{1-\frac{\alpha_1^2}{\alpha_0\alpha_2}}\qquad (0\le q\le 1)$

窄带过程： $0\le q<0.35$

宽带过程： $0.35\le q\le 1$

2.线性体系随机振动

2.1单自由度体系

- 响应卷积方程

$$v_r = \int_{-\infty}^{t} p_r(\tau) h(t-\tau) d\tau \quad (r = 1, 2, \cdots, \infty)$$

- 响应均值

$$E[v(t)] = E\left[\int_{-\infty}^{t} p(\tau) h(t-\tau) d\tau\right] = \int_{-\infty}^{t} E[p(\tau)] h(t-\tau) d\tau = 0$$

- 响应相关函数

$$E[v(t) v(t+\tau)] = E\left[\int_{-\infty}^{t} p(\theta_1) h(t-\theta_1) d\theta_1 \int_{-\infty}^{t} p(\theta_2) h(t-\theta_2) d\theta_2\right]$$

$$= \int_0^{\infty}\int_0^{\infty} E[p(t-u_1) p(t+\tau-u_2)] h(u_1) h(u_2) du_1 du_2$$

$$= \int^{\infty}\int^{\infty} R_p(t-u_2+u_1) h(u_1) h(u_2) du_1 du_2$$

2.1单自由度体系(续)

- 响应谱密度

$$S_v(\varpi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_v(\tau)\exp(-i\varpi\tau)d\tau$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_0^{\infty}\int_0^{\infty}R_p(\tau-u_2+u_1)h(u_1)h(u_2)du_1du_2\right]\exp(-i\varpi\tau)d\tau$$

$$=\frac{1}{2\pi}\lim_{s\to\infty}\left[\int_0^{s}h(u_1)\exp(i\varpi u_1)du_1\int_0^{s}h(u_2)\exp(-i\omega u_2)du_2\right.$$

$$\left.\times\int_{-s+u_1-u_2}^{s+u_1-u_2}R_p(\theta)\exp(-i\varpi\theta)d\theta\right]$$

$$=H(-i\varpi)H(i\varpi)S_p(\varpi)$$

$$=\left|H(i\varpi)\right|^2S_p(\varpi)$$

- 窄带响应过程

包络上限为Rayleigh分布、 $p(\hat{v})=\frac{\hat{v}}{\sigma_v^2}\exp\left(-\frac{1}{2}\frac{\hat{v}^2}{\sigma_v^2}\right)$

2.线性体系随机振动(续)

2.2 多自由度体系

➢振动方程分解

$$\ddot{Y}_n(t)+2\omega_n\xi_n\dot{Y}_n(t)+\omega_n^2Y_n(t)=\frac{P_n(t)}{M_n}\quad (n=1,2,\cdots)$$

$$Z(t)=\sum_n B_nY_n(t)$$

➢响应相关函数

$$R_z(\tau)=E[Z(t)Z(t+\tau)]$$

$$=E\left[\sum_m\sum_n B_mB_nY_m(t)Y_n(t+\tau)\right]$$

$$=E\left[\sum_m\sum_n\int_{-\infty}^{t}\int_{-\infty}^{t+\tau}B_mB_nP_m(\theta_1)P_n(\theta_2)h_m(t-\theta_1)h_n(t+\tau-\theta_2)d\theta_1d\theta_2\right]$$

$$R_Z(\tau)=\sum_m\sum_n R_{z_mz_n}(\tau)\quad (CQC方法)$$

$$R_Z(\tau)\doteq\sum R_{z_mz_m}(\tau)\quad (SRSS\ 方法)$$

2.2 多自由度体系(续)

➢响应谱密度

$$S_z(\bar{\varpi})=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_z(\tau)\exp(-i\bar{\varpi}\tau)d\tau$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\sum_m\sum_n\int_0^{\infty}\int_0^{\infty}B_mB_nR_{P_mP_n}(\tau-u_2+u_1)h_m(u_1)h_n(u_1)du_1du_2\right\}\exp(-i\bar{\varpi}\tau)d$$

$$=\frac{1}{2\pi}\sum_m\sum_n B_mB_n\left[\lim_{T\to\infty}\int_{-T}^{T}h_m(u_1)\exp(-i\bar{\varpi}u_1)du_1\right.$$

$$\left.\times\int_{-T}^{T}h_m(u_2)\exp(i\bar{\varpi}u_2)du_2\int_{-T-u_2+u_1}^{T-u_2+u_1}R_{P_mP_n}\exp(-i\bar{\varpi}r)dr\right]$$

$$S_z(\bar{\varpi})=\sum_m\sum_n S_{z_mz_n}(\bar{\varpi})\equiv\sum_m\sum_n B_mB_nH_m(-i\bar{\varpi})H_n(i\bar{\varpi})S_{p_mp_n}(\bar{\varpi})$$

$$S_z(\bar{\varpi})\doteq\sum_m S_{z_mz_m}(\bar{\varpi})\equiv\sum_m B_m{}^2\left|H_n(i\bar{\varpi})\right|^2S_{p_mp_n}(\bar{\varpi})$$

3.非线性体系随机振动

3.1 前进方程和后退方程

- 振动方程：$\ddot{x}(t)+2\xi_0\omega_0\dot{x}(t)+\omega_0^2x(t)=f(t)$

令：$x_1=x,\quad x_2=\dot{x}$

则：$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\omega_0^2 & -2\xi_0\omega_0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ f(t)\end{bmatrix}$$

或：$$\{\dot{\underline{\overline{X}}}(t)\}=[N]\{\underline{\overline{X}}(t)\}+[G]\{F(t)\}$$

式中：$$\{\underline{\overline{X}}(t)\}=\begin{Bmatrix}x(t)\\ \dot{x}(t)\end{Bmatrix}=\begin{Bmatrix}x_1\\ x_2\end{Bmatrix},\qquad \{F(t)\}=\begin{bmatrix}0\\ f(t)\end{bmatrix}$$

$$[N]=\begin{bmatrix}0 & 1\\ -\omega_0^2 & -2\xi_0\omega_0\end{bmatrix}$$

$$[G]=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=[I]$$

3.1 前进方程和后退方程(续)

- 随机微分方程(Random Differential Equation)

$$\left\{\dot{\underline{X}}(t)\right\}=\left\{h\left(\underline{X},t\right)\right\}+\left[G\left(\underline{X},t\right)\right]\left\{F(t)\right\}$$

- 伊藤方程(Stochastic Differential Equation)

当$\{F(t)\}$为白噪声时，称为伊藤方程，且解为$Markov$过程

- 转移概率密度

前进方程：$$\frac{\partial f\left(\overline{X},t\middle|\overline{X}_0,t_0\right)}{\partial t}=-\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left[a_j f\right]+\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial^2}{\partial x_i\partial x_j}\left[b_{ij} f\right]$$

后退方程：$$\frac{\partial f\left(\overline{X},t\middle|\overline{X}_o,t_o\right)}{\partial t_o}=-\sum_{j=1}^{2}\frac{\partial f}{\partial x_{oj}}a_j-\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial^2 f}{\partial x_{oi}\partial x_{oj}}b_{ij}$$

3.1 前进方程和后退方程(续)

- 导出矩

一阶导出矩： $\{a_j\}=[N]\{\overline{X}(t)\}=\begin{Bmatrix} x_2 \\ -\omega_0^2 x_1 - 2\xi_0\omega_0 x_2 \end{Bmatrix}$

二阶导出矩： $[b_{ij}]=[2GDG^T]=\begin{bmatrix} 0 & 0 \\ 0 & 2\pi S_0 \end{bmatrix}$

式中： D为白噪声过程$F(t)$的协方差矩阵

- 初始条件

$$f\left(\overline{X},t\middle|\overline{X}_0,t_0\right)=\prod_{j=1}^{2}\delta\left(x_j-x_{j0}\right)$$

- 二维高斯分布

满足前进方程和后退方程的转移概率为二维高斯分布

3.1 前进方程和后退方程(续)

- 统计参数

均值：$M_1 = \frac{\alpha}{\overline{\omega}}\left[x_{10}\left(\overline{\omega}\cos\overline{\omega}t + \xi\omega_0\sin\overline{\omega}t\right) + \omega_0 x_{20}\sin\overline{\omega}t\right]$

$$M_2 = \frac{\alpha}{\overline{\omega}}\left[\omega_0\left\{x_{20}\left(\overline{\omega}\cos\overline{\omega}t - \xi\omega_0\sin\overline{\omega}t\right) - \omega_0 x_{10}\sin\overline{\omega}t\right\}\right]$$

协方差：$\Lambda_{11} = \frac{\beta\sigma_x^2}{\overline{\omega}^2}\left(\overline{\omega}^2 + 2\xi^2\omega_0^2\sin\overline{\omega}t + 2\xi\overline{\omega}\omega_0\sin\overline{\omega}t\cos\overline{\omega}t\right)$

$$\Lambda_{12} = \Lambda_{21} = \frac{\pi S_0}{\overline{\omega}^2}\alpha^2\sin^2\overline{\omega}t$$

$$\Lambda_{22} = \frac{\beta\sigma_x^2}{1-\xi^2}\left(\overline{\omega}^2 + 2\xi^2\omega_0^2\sin\overline{\omega}t - 2\xi\overline{\omega}\omega_0\sin\overline{\omega}t\cos\overline{\omega}t\right)$$

参数：$\overline{\omega} = \sqrt{1-\xi^2}\,\omega_0,\quad \alpha = \alpha(t) = \exp(-\xi\omega t)$

$$\beta = \beta(t) = 1 - \exp(-2\xi\omega_0 t)$$

$$\sigma_x^2 = \frac{\pi S_0}{2\xi\omega_o^3}$$

3.非线性体系随机振动(续)

3.2 FPK法应用

- 非线性振动方程

$$m\ddot{x}(t)+g(x,\dot{x})=f(t)$$

$$\{\dot{\underline{X}}(t)\}=\{F(t)\}+[G]\{F(t)\}$$

$$\{\underline{X}(t)\}=\{x(t),\dot{x}(t)\}^T=\{x_1,x_2\}^T$$

$$\{F(t)\}=\{0,f(t)\}^T$$

$$\{N(t)\}=\{x_2,-g(x_1,x_2)\}^T$$

$$[G]=[I]$$

3.2 FPK法应用(续)

- 导出矩

一阶导出矩： $\{a_j\}=[N(\underline{\overline{X}},t)]=\begin{Bmatrix} x_2 \\ -g(x_1,x_2) \end{Bmatrix}$

二阶导出矩： $[b_{ij}]=[2GDG^T]=\begin{bmatrix} 0 & 0 \\ 0 & I(t) \end{bmatrix}$

- 求解方程

$$\frac{\partial f(\underline{\overline{X}},t|\underline{\overline{X}}_0,t_0)}{\partial t}=-x_2\frac{\partial f}{\partial x_1}+\frac{\partial}{\partial x_2}[fg(x_1,x_2)]+\frac{I(t)}{2}\frac{\partial^2 f}{\partial x_2^2}$$

式中：$I(t)$—随机荷载强度函数，即$E[f(t)f(t+\tau)]=I(t)\delta(\tau)$

当$f(t)$是高斯白噪声过程时，且$\frac{\partial f}{\partial t}=0$时，才有解

- 响应推理：

荷载高斯白噪声+刚度非线性→位移与速度独立

→位移非高斯变量+速度高斯变量

3.非线性体系随机振动(续)

3.3 等效线性化法

- 线性方程： $m\ddot{x}(t)+C_e\dot{x}(t)+K_ex(t)=f(t)$
- 非线性方程： $m\ddot{x}(t)+g(x,\dot{x})=f(t)$
- 两式之差： $e=g(x,\dot{x})-C_e\dot{x}(t)-K_ex(t)$
- 令e的均方值最小求参数C_e和K_e

$$\begin{cases}\dfrac{\partial E(e^2)}{\partial C_e}=2E\left[e\dfrac{\partial e}{\partial C_e}\right]=2E(e\dot{x})=0\\[2ex]\dfrac{\partial E(e^2)}{\partial K_e}=2E\left[e\dfrac{\partial e}{\partial K_e}\right]=2E(ex)=0\end{cases}$$

$$\begin{cases}C_e=\dfrac{E(x^2)E(\dot{x}g)-E(\dot{x}^2)E(xg)}{E(x^2)E(\dot{x}^2)-E^2(x\dot{x})}\overset{\text{平稳高斯}}{=\!=\!=\!=}\dfrac{E(\dot{x}g)}{E(\dot{x}^2)}\\[2ex]K_e=\dfrac{E(\dot{x}^2)E(xg)-E(x^2)E(\dot{x}g)}{E(x^2)E(\dot{x}^2)-E^2(x\dot{x})}\overset{\text{平稳高斯}}{=\!=\!=\!=}\dfrac{E(xg)}{E(x^2)}\end{cases}$$

3.非线性体系随机振动(续)

3.4 奇异摄动法

- 非线性恢复力：$\ddot{x}+2\xi\dot{x}+\omega_0^2\left[x+\varepsilon g(x)\right]=f(t)$
- 常系数解：　$x(t)=x_0(t)+\varepsilon x_1(t)+\varepsilon^2 x_2(t)+\cdots$
- $g(x)$泰勒展开：

$$g(x)=g(x_0)+\frac{g'(x_0)}{1!}\left[\varepsilon x_1(t)+\varepsilon^2 x_2(t)+\cdots\right]+\frac{g''(x_0)}{2!}\left[\varepsilon x_1(t)+\varepsilon^2 x_2(t)+\cdots\right]^2+\cdots$$

- 同次项合并

$$\ddot{x}_0+2\xi\dot{x}_0+\omega_0^2 x_0=f(t)$$

$$\ddot{x}_1+2\xi\dot{x}_1+\omega_0^2 x_1=-\omega_0^2 g(x_0)$$

$$\ddot{x}_2+2\xi\dot{x}_2+\omega_0^2 x_2=-\omega_0^2 x_1 g'(x_0)$$

………

3.4 奇异摄动法(续)

- 摄动解

$$x(t)=x_0(t)+\varepsilon x_1(t)$$

$$x_0(t)=\int_0^\infty h(\tau)f(t-\tau)d\tau$$

$$x_1(t)=\int_0^\infty h(\tau)\{-\omega_0^2 g[x_0(t-\tau)]\}d\tau$$

$$=-\omega_0^2\int_0^\infty h(\tau)g[x_0(t-\tau)]d\tau$$

小 结

功率谱分析方法应用—功率谱与相关函数关系

线性体系随机振动—谱分析方法

非线性体系随机振动—转移概率密度

- FPK方法—位移非高斯、速度高斯
- 等效线性化法—C_e和K_e等效线性
- 奇异摄动法—非线性恢复力

下周同一时间再见!